

Pairs of commuting Hamiltonians, quadratic in momenta

V.G. Marikhin ¹ and V.V. Sokolov ¹

¹ L.D. Landau Institute for Theoretical Physics RAS, Moscow, Russia

In the case of two degree system the pairs of quadratic in momenta Hamiltonians commuting according the standard Poisson bracket are considered. The new many-parametrical families of such pairs are founded. The universal method of constructing the full solution of Hamilton - Jacobi equation in terms of integrals on some algebraic curve is proposed. For some examples this curve is non-hyperelliptic covering over the elliptic curve.

MSC numbers: 17B80, 17B63, 32L81, 14H70

Address: Landau Institute for Theoretical Physics RAS, Kosygina st.2 ,Moscow, Russia, 119334,

E-mail: mvg@itp.ac.ru, sokolov@itp.ac.ru

1 Pairs of quadratic hamiltonians

In papers [1, 2, 3, 4, 5, 6, 7] the problem of the commuting pairs of Hamiltonians quadratic in momenta was considered.

Consider pair of Hamiltonians in the form

$$H = ap_1^2 + 2bp_1p_2 + cp_2^2 + dp_1 + ep_2 + f, \quad (1.1)$$

$$K = Ap_1^2 + 2Bp_1p_2 + Cp_2^2 + Dp_1 + Ep_2 + F, \quad (1.2)$$

commuting with respect to standart poisson bracket $\{p_\alpha, q_\beta\} = \delta_{\alpha\beta}$. The coefficients in formulas (1.1),(1.2) - some (locally) analitical functions of the variables q_1, q_2 .

Theorem 1. *Any pairs of commuting Hamiltonians (1.1)-(1.2) can be canonically transformed by*

$$\hat{P}_1 = P_1 + \frac{\partial F(s_1, s_2)}{\partial s_1}, \quad \hat{P}_2 = P_2 + \frac{\partial F(s_1, s_2)}{\partial s_2}$$

to the pair of the form

$$H = \frac{U_1 - U_2}{s_1 - s_2}, \quad K = \frac{s_2 U_1 - s_1 U_2}{s_1 - s_2}, \quad (1.3)$$

where

$$U_1 = S_1(s_1)P_1^2 + \frac{\sqrt{S_1(s_1)S_2(s_2)}Z_{s_1}}{(s_1 - s_2)}P_2 - \frac{S_1(s_1)Z_{s_1}^2}{4(s_1 - s_2)^2} + V_1(s_1, s_2), \quad (1.4)$$

$$U_2 = S_2(s_2)P_2^2 - \frac{\sqrt{S_1(s_1)S_2(s_2)}Z_{s_2}}{(s_1 - s_2)}P_1 - \frac{S_2(s_2)Z_{s_2}^2}{4(s_2 - s_1)^2} + V_2(s_1, s_2),$$

$$V_1 = \frac{1}{2}\sqrt{S_1(s_1)}\partial_{q_1}\left(\sqrt{S_1(s_1)}\frac{Z_{s_1}^2}{s_1 - s_2}\right) + f_1(s_1), \quad (1.5)$$

$$V_2 = \frac{1}{2}\sqrt{S_2(s_2)}\partial_{q_2}\left(\sqrt{S_2(s_2)}\frac{Z_{s_2}^2}{s_2 - s_1}\right) + f_2(s_2)$$

for some functions $Z(s_1, s_2)$, $S_i(s_i)$ and $f_i(s_i)$. Poisson bracket $\{H, K\}$ equals to zero if and only if

$$Z_{s_1, s_2} = \frac{Z_{s_1} - Z_{s_2}}{2(s_2 - s_1)} \quad (1.6)$$

and

$$\left(Z_{s_1}\frac{\partial}{\partial s_2} - Z_{s_2}\frac{\partial}{\partial s_1}\right)\left(\frac{V_1 - V_2}{s_1 - s_2}\right) = 0. \quad (1.7)$$

Proof We introduce new coordinates s_1, s_2 , such that the quadratic parts of H, K (1.1,1.2) are diagonal: Let s_1, s_2 be the roots of equations

$$\Phi(s, q_1, q_2) = (B - bs)^2 - (A - as)(C - cs) = 0, \quad (1.8)$$

Then the canonical transformation

$$(q_1, q_2, p_1, p_2) \rightarrow (s_1, s_2, P_1, P_2) : p_1 = -\left(\frac{\Phi_{q_1}^1}{\Phi_{s_1}^1}P_1 + \frac{\Phi_{q_1}^2}{\Phi_{s_2}^2}P_2\right), \quad p_2 = -\left(\frac{\Phi_{q_2}^1}{\Phi_{s_1}^1}P_1 + \frac{\Phi_{q_2}^2}{\Phi_{s_2}^2}P_2\right), \quad (1.9)$$

where $\Phi^i = \Phi(s_i, q_1, q_2)$ under conditions $\{H, K\} = 0$ transforms pairs (1.1), (1.2) to the form

$$H = \frac{U_1 - U_2}{s_1 - s_2}, \quad K = \frac{s_2 U_1 - s_1 U_2}{s_1 - s_2}, \quad (1.10)$$

where

$$U_1 = S_1(s_1)P_1^2 + \tilde{d}P_1 + \tilde{e}P_2 + \tilde{f}, \quad U_2 = S_2(s_1)P_2^2 + \tilde{D}P_1 + \tilde{E}P_2 + \tilde{F}, \quad (1.11)$$

where

$$S_i(s_i) = \frac{1}{(\Phi_{q_i}^i)^2} ((as_i - A)(\Phi_{q_1}^i)^2 + 2(bs_i - B)\Phi_{q_1}^i \Phi_{q_2}^i + (cs_i - C)(\Phi_{q_2}^i)^2) \quad (1.12)$$

We calculate a Poisson bracket between H and K. Then the coefficient of P_1^2, P_2^2, P_1P_2 equal to zero iff

$$\tilde{d} = 2S_1(s_1)\frac{\partial F(s_1, s_2)}{\partial s_1}, \quad \tilde{e} = \frac{\sqrt{S_1(s_1)S_2(s_2)}Z_{s_1}}{(s_1 - s_2)}, \quad \tilde{D} = -\frac{\sqrt{S_1(s_1)S_2(s_2)}Z_{s_2}}{(s_1 - s_2)}, \quad \tilde{E} = 2S_2(s_2)\frac{\partial F(s_1, s_2)}{\partial s_2}$$

where $Z(s_1, s_2), F(s_1, s_2)$ - some functions. and

$$Z_{s_1, s_2} = \frac{Z_{s_1} - Z_{s_2}}{2(s_2 - s_1)} \quad (1.13)$$

We apply the canonical transformation

$$\hat{P}_1 = P_1 + \frac{\partial F(s_1, s_2)}{\partial s_1}, \quad \hat{P}_2 = P_2 + \frac{\partial F(s_1, s_2)}{\partial s_2}$$

to equate \tilde{d}, \tilde{E} to zero. Then the coefficient of P_1, P_2 equal to zero iff U_1, U_2 have the form as in formulation of Theorem 1. And finally the free coefficient in Poisson bracket equals to zero iff the equation (1.7) of the Theorem 1 is fulfilled just as expected.

The general analytical solution of Euler - Darboux equation (1.6) has near the line of singularities $x = y$ the following expansion:

$$Z(x, y) = A + \ln(x - y) B, \quad A = \sum_0^\infty a_i(x + y)(x - y)^{2i}, \quad B = \sum_0^\infty b_i(x + y)(x - y)^{2i},$$

where a_0 and a_1 - some function. The other coefficients can be expressed by these two functions and their derivatives. For example, $b_0 = \frac{1}{2}a_0''$.

We insert this expansion into (1.7) to obtain $B = 0$. It is easy to check that any solution of the equation (1.6) with $B = 0$ has the form

$$Z(x, y) = z_0 + \delta(x + y) + (x - y)^2 \sum_{k=0}^\infty \frac{g^{(2k)}(x + y)}{2^{(2k)}k!(k + 1)!} (x - y)^{2k}, \quad (1.14)$$

where $g(x)$ - some function and z_0, δ - some constants. We call the function $g(x)$ as *generating function* for (1.14). Without the loss of generality we choose $z_0 = 0$. The parameter δ , is very important for classification of hamiltonians from Theorem 1.

We find all the functions Z , corresponding the rational generating functions g . Choosing $g(x) = x^n$, we obtain the infinite set of polynomial solutions $Z^{(n)}$ for (1.6). In particular

$$\begin{aligned} g(x) = 1 &\iff Z^{(0)}(x, y) = (x - y)^2 \\ g(x) = x &\iff Z^{(1)}(x, y) = (x + y)(x - y)^2, \\ g(x) = x^2 &\iff Z^{(2)}(x, y) = \frac{1}{4} ((x - y)^2 + 4(x + y)^2) (x - y)^2. \end{aligned}$$

All set can be obtained by using 'creating' operator

$$x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - \frac{1}{2}(x + y),$$

acting on $Z^{(0)}$. The rational functions $g(x) = (x - \mu)^{-n}$ create one more class of exact solution of equation (1.6). For example

$$g_\mu(x) = \frac{1}{4} \frac{1}{x - 2\mu} \iff Z_\mu(x, y) = \sqrt{(\mu - x)(\mu - y)} + \frac{1}{2}(x + y) - \mu.$$

The solution corresponding the poles of order $n \geq 2$, can be obtained by differentiating the last formula by parameter μ . Because function Z is linear by g we obtained the solution Z with rational generating function $g(x) = \sum_i c_i x^i + \sum_{i,j} d_{ij} (x - \mu_i)^{-j}$.

Hypothesis 1. For all Hamiltonians (1.3)-(1.7) generating function g is rational and has the form $g(x) = \frac{P(x)}{S(x)}$, where P и S - some polynomials with $\deg P < 5$, $\deg S < 6$.

In papers [5, 6] the following solution of the system (1.6), (1.7) was considered:

$$\begin{aligned} Z(x, y) &= x + y, \quad S_1(x) = S_2(x) = \sum_{i=0}^6 c_i x^i, \\ f_1(x) = f_2(x) &= -\frac{3}{4} c_6 x^4 - \frac{1}{2} c_5 x^3 + \sum_{i=0}^2 k_i x^i, \end{aligned}$$

where c_i, k_i - some constants. A very important fact is that Clebsch top and $so(4)$ -Schottky-Manakov top [8, 9, 10] are the particular cases of this model [6]. In paper [6] a full solution of Hamilton - Jacobi equation of this model was obtained in the form of some kind of separation of variables on a non-hyperelliptic curve of genus 4.

2 Universal solution of Hamilton-Jacobi equation

Let H and K have the form (1.3)-(1.5). Consider system $H = e_1$, $K = e_2$, where e_i - some constants. Let $p_1 = F_1(x, y)$, $p_2 = F_2(x, y)$ - be its solution. We use short notation x и y corresponding q_1 и q_2 . Jacobi's lemma gives that if $\{H, K\} = 0$, then $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. To find an action $S(x, y, e_1, e_2)$, it is enough to solve the following system

$$\frac{\partial}{\partial x} S = F_1, \quad \frac{\partial}{\partial y} S = F_2.$$

We rewrite the system $H = e_1$, $K = e_2$ in the form

$$p_1^2 + ap_2 + b = 0, \quad p_2^2 + Ap_1 + B = 0, \quad (2.15)$$

where

$$\begin{aligned} a &= \frac{Z_x}{x-y} \sqrt{\frac{S_2(y)}{S_1(x)}}, & A &= -\frac{Z_y}{x-y} \sqrt{\frac{S_1(x)}{S_2(y)}} \\ b &= -\frac{Z_x^2}{4(x-y)^2} + \frac{V_1 - e_1x + e_2}{S_1(x)}, & B &= -\frac{Z_y^2}{4(x-y)^2} + \frac{V_2 - e_1y + e_2}{S_2(y)}. \end{aligned}$$

It easy to find that

$$2b_y + Aa_x + 2aA_x = 0, \quad 2Aa_y + aA_y + 2B_x = 0. \quad (2.16)$$

Using (1.6) and (1.7), it is easy to obtain the following identity

$$Ab_x - aB_y + 2A_xb - 2a_yB = 0. \quad (2.17)$$

Using a standard technique of Lagrange resolvents (see f.e. [11]), we rewrite system (2.15) to a system

$$uv = \frac{1}{4}aA, \quad (2.18)$$

$$Au^3 + 4\frac{b}{a}u^2v - 4\frac{B}{A}uv^2 - av^3 = 0, \quad (2.19)$$

that is equivalent to the cubic equation on u^2 . Let (u_k, v_k) , $k = 1, 2, 3$ be the solutions of (2.18), (2.19) such that

$$\begin{aligned} u_1^2 + u_2^2 + u_3^2 &= -b, & v_1^2 + v_2^2 + v_3^2 &= -B \\ u_1u_2u_3 &= -\frac{1}{8}a^2A, & v_1v_2v_3 &= -\frac{1}{8}A^2a. \end{aligned}$$

Then, formulas

$$\begin{aligned} p_1 &= u_1 + u_2 + u_3, & p_2 &= v_1 + v_2 + v_3; \\ p_1 &= u_3 - u_1 - u_2, & p_2 &= v_3 - v_1 - v_2; \\ p_1 &= u_2 - u_1 - u_3, & p_2 &= v_2 - v_1 - v_3; \\ p_1 &= u_1 - u_2 - u_3, & p_2 &= v_1 - v_2 - v_3 \end{aligned}$$

define four solutions of (2.15). Consider the first of them.

Lemma 1. *For $i = 1, 2, 3$ following equations are fulfilled $\frac{\partial u_i}{\partial y} = \frac{\partial v_i}{\partial x}$.*

Prove. Differentiating equations (2.18) and (2.19) on x and y , we find u_y and v_x as the functions on u and v . Then expressing v through u , we obtain that $u_y = v_x$ is equivalent to identities (2.16) and (2.17). ■

Lemma 1 means, that in variables u_1, u_2, u_3 we find "particular" separation variables. Really $S = S_1 + S_2 + S_3$, where S is the action, and functions S_i defined from a system

$$\frac{\partial}{\partial x} S_i = u_i, \quad \frac{\partial}{\partial y} S_i = v_i.$$

Let's

$$u = \frac{1}{2} \frac{Z_x}{x-y} \sqrt{\frac{y-\xi}{x-\xi}}, \quad v = -\frac{1}{2} \frac{Z_y}{x-y} \sqrt{\frac{x-\xi}{y-\xi}}.$$

It easy to see that pair (u, v) for all ξ are a solution of (2.18). If Z is a solution of (1.6), then $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. Using this fact we introduce a function $\sigma(x, y, \xi)$ so that

$$\frac{\partial \sigma}{\partial x} = u, \quad \frac{\partial \sigma}{\partial y} = v.$$

In a case of rational function g , corresponding function Z is expressed through quadratic radicals and the function σ can be obtained. Let's $Y = \frac{\partial \sigma}{\partial \xi}$.

After multiplication of expression (2.19) by expression

$$-2 \frac{\sqrt{S_1(x)} \sqrt{S_2(y)} \sqrt{x-\xi} \sqrt{y-\xi} (x-y)}{Z_x Z_y},$$

left side of (2.19) can be written in the form

$$-e_2 + e_1 \xi + \frac{y-\xi}{x-y} \left(V_1 - \frac{S_1(x) Z_x^2}{4(x-\xi)(x-y)} \right) - \frac{x-\xi}{x-y} \left(V_2 + \frac{S_2(y) Z_y^2}{4(y-\xi)(x-y)} \right). \quad (2.20)$$

Proposition 1. *Let the expression (1.6), (1.7) be fulfilled. Then the expression (2.20) is a function of Y and ξ variables only.*

Prove. We assign the function (2.20) as $\Psi(x, y, \xi)$. Consider Jacobian

$$J = \frac{\partial \Psi}{\partial x} \frac{\partial Y}{\partial y} - \frac{\partial \Psi}{\partial y} \frac{\partial Y}{\partial x}.$$

We change $\frac{\partial Y}{\partial y}$ and $\frac{\partial Y}{\partial x}$ to $\frac{\partial v}{\partial \xi}$ and $\frac{\partial u}{\partial \xi}$, respectevily, then Jacobian J equals to zero identically taking into account (1.6), (1.7). ■

Due to Proposition 1, the relation $\Psi(x, y, \xi) = 0$ can be rewritten in the form $\phi(\xi, Y) = 0$. One can find the function ϕ by assuming $y = x$.

Equation $\phi(\xi, Y) = 0$ defines a curve, and the differentials of this curve define the function of action S .

We note $\xi_k(x, y)$, where $k = 1, 2, 3$, the roots of cubic equation $\Psi(x, y, \xi) = 0$.

Theorem 2. *The function of action S has the form*

$$S(x, y) = \sum_{k=1}^3 \left(\sigma(x, y, \xi_k) - \int^{\xi_k} Y(\xi) d\xi \right), \quad (2.21)$$

where $Y(\xi)$ - algebraic function on the curve $\phi(\xi, Y) = 0$.

Prove. We obtain

$$\frac{\partial}{\partial x} S(x, y) = \sum_{k=1}^3 \sigma_x(x, y, \xi_k) + \sum_{k=1}^3 \{ \sigma_\xi(x, y, \xi_k) - Y(\xi_k) \} \xi_{k,x} = \sum_{k=1}^3 u_k = p_1.$$

Analogously

$$\frac{\partial}{\partial y} S(x, y) = p_2. \quad \blacksquare$$

3 Case of cubics

Consider a case when the curve (2.20) can be written in the form $\tilde{\phi}(\xi, \eta) = 0 \Leftrightarrow \phi(\xi, Y)$, so that points $(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)$ lie on a straight line, that equivalent to definition $\eta = \xi a(x, y) + b(x, y)$, its substitution into $\tilde{\phi}$, gives a curve $\Psi(x, y, \xi) = 0$.

Formula (2.20) gives a curve in a new variables $\xi, \eta - e_2 + e_1 \xi + \frac{C_2(\xi, \eta)}{C_1(\xi, \eta)} = 0$, where $C_1(\xi, \eta) \rightarrow 0$ at $x \rightarrow 0$ or $y \rightarrow 0$.

Using reversible curve equation $C_1(\xi, \eta) = 0 \rightarrow \eta = f(\xi)$ using η , we find the expressions for $a(x, y)$, $b(x, y)$

$$a(x, y) = \frac{f(x) - f(y)}{x - y}, \quad b(x, y) = \frac{yf(x) - xf(y)}{x - y}$$

On the other hand the equivalence of the curve $\phi(\xi, Y) = 0$ и $\tilde{\phi}(\xi, \eta) = 0$ gives

$$Y_x \eta_y = Y_y \eta_x \Leftrightarrow u_\xi \eta_y = v_\xi \eta_x \Leftrightarrow (\xi - y) Z_x \eta_y = (x - \xi) Z_y \eta_x,$$

or $Z = Z(a)$, $b_x = -y a_x$, $b_y = -x a_y$.

4 Examples

In this Section we consider all the pairs of Hamiltonians known at the moment (1.3)-(1.7).

4.1 Class 1

For the models of this class

$$S_1 = S_2 = S, \quad f_1 = f_2 = f. \quad (4.22)$$

Theorem 3. *Let*

$$g = \frac{\tilde{G}}{S}, \quad \tilde{G} = G - \frac{\delta}{10} S', \quad f = -\frac{4\tilde{G}^2}{S} - \frac{4\delta}{3} \tilde{G}' - \frac{\delta^2}{12} S'',$$

where

$$S(x) = s_5 x^5 + s_4 x^4 + s_3 x^3 + s_2 x^2 + s_1 x + s_0, \quad G(x) = g_3 x^3 + g_2 x^2 + g_1 x + g_0,$$

where s_i, g_i, δ - some constants. Then functions S , f and function Z , corresponding (see. §1) generation function g , fulfill the systems (1.6), (1.7).

Remark. Parameter δ from Theorem 3 coincides with parameter δ from (1.14). Consider the case $\delta = 0$ in the formula (1.14), Then all pairs of Hamiltonians (1.3)-(1.7), (4.22), that fulfill this condition are described by Theorem 3.

Consider a general case

$$S(x) = s_5(x - \mu_1)(x - \mu_2)(x - \mu_3)(x - \mu_4)(x - \mu_5),$$

where $s_5 \neq 0$ and all roots μ_i of polynomial S are distinct. then the function Z has the form

$$Z(x, y) = \sum_{i=1}^5 \nu_i \sqrt{(\mu_i - x)(\mu_i - y)}, \quad (4.23)$$

where ν_i - some constants. Coefficients g_i and δ are expressed through constants ν_j from (1.14). For example, $2\delta = -\sum \nu_i$. Function f is defined by

$$f(x) = -\frac{1}{16} \sum_{i=1}^5 \nu_i^2 \frac{S'(\mu_i)}{x - \mu_i} + k_1 x + k_0,$$

where k_1, k_0 - some constants.

Calculation for a function (4.23) gives

$$\sigma(x, y, \xi) = -\frac{1}{2} \sum_{i=1}^5 \nu_i \log \frac{\sqrt{x - \xi} \sqrt{y - \mu_i} + \sqrt{y - \xi} \sqrt{x - \mu_i}}{\sqrt{x - y} \sqrt{\mu_i - \xi}}, \quad (4.24)$$

$$Y = \frac{1}{4} \sum_{i=1}^N \nu_i \frac{\sqrt{(x - \mu_i)(y - \mu_i)}}{(\xi - \mu_i) \sqrt{(x - \xi)(y - \xi)}}.$$

Algebraic curve has the form of hyperelliptic curve of genus = 2

$$\phi(Y, \xi) = S(\xi)Y^2 + f(\xi) - \xi e_1 + e_2 = 0$$

Steklov top on $so(4)$ [12] is a particular case of Theorem 3.

4.2 Class 2

Functions Z for the models of this class are the special cases of the functions Z of Class 1. But this Class contains much more parameters than Theorem 3.

Such functions Z can be defined as the solutions of system

$$Z_{xy} = \frac{Z_x - Z_y}{2(y - x)} = \frac{1}{3} U(Z) Z_x Z_y, \quad (4.25)$$

where U - some functions of one variable.

Remark. It easy to see that this class of solutions of Euler - Darboux equation $Z_{xy} = \frac{Z_x - Z_y}{2(y - x)}$ coincide with the class of solutions of the form

$$Z = F\left(\frac{h(x) - h(y)}{x - y}\right),$$

where F and h - some functions of one variable and $U = F''/F'^2$.

Lemma. *The system (4.25) is compatible if and only if*

$$U = \frac{3}{2} \frac{B'}{B}, \quad B(Z) = b_2 Z^2 + b_1 Z + b_0,$$

where b_i - some constants.

In a case $\deg B = 2$

$$Z(x, y) = \sqrt{(x - \mu_1)(y - \mu_1)} + \sqrt{(x - \mu_2)(y - \mu_2)}, \quad (4.26)$$

where $b_2 = 1, \quad b_1 = 0, \quad b_0 = -(\mu_1 - \mu_2)^2$.

If $\deg B = 1$, then

$$Z(x, y) = \sqrt{xy} + \frac{1}{2}(x + y), \quad (4.27)$$

$b_1 = 1, b_2 = b_0 = 0$.

If $\deg B = 0$, then

$$Z(x, y) = x + y. \quad (4.28)$$

1. Consider function Z of the form (4.26). Then

$$S(x) = (x - \mu_1)(x - \mu_2)P(x) + (x - \mu_1)^{3/2}(x - \mu_2)^{3/2}Q(x), \quad \deg P \leq 3, \deg Q \leq 2,$$

и

$$\begin{aligned} f(x) = f_0 + f_1 x + k_2(x - \mu_1)^{1/2}(x - \mu_2)^{1/2} + \frac{(\mu_2 - \mu_1)}{16} \left\{ \frac{P(\mu_1)}{x - \mu_1} - \frac{P(\mu_2)}{x - \mu_2} \right\} \\ + \frac{(\mu_2 - \mu_1)}{32} (x - \mu_1)^{1/2}(x - \mu_2)^{1/2} \left\{ \frac{Q(\mu_1)}{x - \mu_1} - \frac{Q(\mu_2)}{x - \mu_2} \right\}. \end{aligned}$$

In a case when $Q = 0, k_2 = 0$, These formulas coincide with corresponding formulas of Class

1. The functions σ, Y are defined the same formula (4.24) as for Class 1 :

$$\sigma(x, y, \xi) = -\frac{1}{2} \sum_{i=1}^2 \log \frac{\sqrt{x - \xi} \sqrt{y - \mu_i} + \sqrt{y - \xi} \sqrt{x - \mu_i}}{\sqrt{x - y} \sqrt{\mu_i - \xi}}, \quad Y = \frac{1}{4} \sum_{i=1}^2 \frac{\sqrt{(x - \mu_i)(y - \mu_i)}}{(\xi - \mu_i) \sqrt{(x - \xi)(y - \xi)}}.$$

Algebraic curve in this case has the form

$$[S_R(\xi) + \eta S_I(\xi)]Y^2 - [k_R(\xi) + \eta k_I(\xi)] = 0, \quad (4.29)$$

where

$$S_R(x) = (x - \mu_1)(x - \mu_2)P(x), \quad S_I(x) = (x - \mu_1)(x - \mu_2)Q(x),$$

$$k_R(x) = -e_2 + e_1 x - f_0 - f_1 x - \frac{(\mu_2 - \mu_1)}{16} \left\{ \frac{P(\mu_1)}{x - \mu_1} - \frac{P(\mu_2)}{x - \mu_2} \right\},$$

$$k_I(x) = k_2 - \frac{1}{32}(\mu_1 - \mu_2)^2 - \frac{1}{16}(\mu_1 - \mu_2) \left\{ \frac{Q(\mu_1)}{x - \mu_1} - \frac{Q(\mu_2)}{x - \mu_2} \right\},$$

$$\frac{1}{\eta} = \frac{1}{\sqrt{\xi - \mu_1} \sqrt{\xi - \mu_2}} \sqrt{1 - \frac{(\mu_1 - \mu_2)^2}{16(\xi - \mu_1)^2(\xi - \mu_2)^2 Y^2}}.$$

Expressing Y as a function of (ξ, η) and substituting to (4.29), we obtain 10-parameter cubic in (ξ, η) , variables.

So in a general case the curve $\phi(Y, \xi) = 0$, is a covering over an elliptic curve. We obtain

$$\eta = \frac{\xi - \mu_1}{\frac{\sqrt{x-\mu_1}}{\sqrt{x-\mu_2}} + \frac{\sqrt{y-\mu_1}}{\sqrt{y-\mu_2}}} + \frac{\xi - \mu_2}{\frac{\sqrt{x-\mu_2}}{\sqrt{x-\mu_1}} + \frac{\sqrt{y-\mu_2}}{\sqrt{y-\mu_1}}},$$

therefore points $(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)$ lie on a straight line.

2. For the function Z of the form (4.27) we have

$$S(x) = xP(x) + x^{3/2}Q(x), \quad \deg P \leq 3, \quad \deg Q \leq 2,$$

$$f(x) = -\frac{1}{16x}P(x) - \frac{1}{32\sqrt{x}}Q(x) + f_1x + f_q\sqrt{x} + f_0.$$

The function Y is defined by $Y = \frac{\xi + \sqrt{x}\sqrt{y}}{4\xi\sqrt{x-\xi}\sqrt{y-\xi}}$. The curve in this case can be written in the form (4.29), where

$$S_R(x) = xP(x), \quad S_I(x) = xQ(x),$$

$$k_R(x) = -e_2 + e_1x - f_0 - f_1x + \frac{1}{16x}P(x), \quad k_I(x) = \frac{1}{16x}Q(x) - f_q,$$

$$\eta = \frac{4Y\xi^{3/2}}{\sqrt{16Y^2\xi^2 - 1}}.$$

In (ξ, η) variables it also has the form of arbitrary cubic. Formula $\eta = \frac{\xi + \sqrt{xy}}{\sqrt{x} + \sqrt{y}}$ gives the fact that points $(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)$ lie on a straight line.

3. For the function Z , given by (4.28), we obtain

$$S(x) = s_6x^6 + s_5x^5 + s_4x^4 + s_3x^3 + s_2x^2 + s_1x + s_0,$$

$$f(x) = -\frac{1}{40}S''(x) - \frac{1}{32\sqrt{x}}Q(x) + f_2x^2 + f_1x + f_0.$$

In this case $Y = \frac{1}{2\sqrt{x-\xi}\sqrt{y-\xi}}$. Algebraic curve

$$S(\xi)Y^6 - F(\xi)Y^4 - \left(\frac{1}{8}F''(\xi) + \frac{7}{1920}S^{IV}(\xi) - \frac{k_2}{2}\right)Y^2 - \frac{s_6}{64} = 0, \quad F(\xi) = -e_2 + e_1\xi - f(\xi)$$

and in (ξ, η) , variables where $\eta = \xi^2 - \frac{1}{4Y^2}$, has the form of arbitrary cubic. Because $\eta = \xi(x+y) - xy$, points $(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)$ belong to a straight line.

4.3 Class 3

We introduce 'non-symmetrical Hamiltonian (1.3)-(1.7) if $S_1(x) \neq S_2(x)$, or $f_1(x) \neq f_2(x)$.

Theorem 4. [7] *In non-symmetrical case the functions Z, S_i, f_i is the solutions of (1.6), (1.7) if and only if*

$$\delta = 0, \quad g = \frac{1}{H}, \quad S_{1,2} = W H \pm M H^{3/2}, \quad f_{1,2} = -\frac{4W}{H} \mp 2M H^{-1/2} \pm a H^{1/2},$$

where g - generation function of Z ,

$$W(x) = w_3 x^3 + w_2 x^2 + w_1 x + w_0, \quad H(x) = h_2 x^2 + h_1 x + h_0,$$

$$M(x) = m_2 x^2 + m_1 x + m_0.$$

Here w_i, h_i, m_i, a - some constants.

Consider the general case $H(x) = (x - \mu_1)(x - \mu_2)$. Algebraic curve in this case is defined by

$$\begin{aligned} \Psi(\xi, Y) = -e_2 + e_1 \xi - \frac{R W(\xi)}{2(\xi - \mu_1)(\xi - \mu_2)(\mu_2 - \mu_1)} + \\ 4M(\xi) \sqrt{2} Y \frac{\sqrt{\xi - \mu_1} \sqrt{\xi - \mu_2}}{(\mu_2 - \mu_1)^{3/2}} \sqrt{R} + 8b \sqrt{2} Y \frac{(\xi - \mu_1)^{3/2} (\xi - \mu_2)^{3/2}}{\sqrt{R} \sqrt{\mu_2 - \mu_1}} = 0, \end{aligned} \quad (4.30)$$

where

$$Y = \frac{\sqrt{(x - \mu_1)(y - \mu_1)}}{(\xi - \mu_1) \sqrt{(x - \xi)(y - \xi)}} - \frac{\sqrt{(x - \mu_2)(y - \mu_2)}}{(\xi - \mu_2) \sqrt{(x - \xi)(y - \xi)}}, \quad R = 16(\xi - \mu_1)^2 (\xi - \mu_2)^2 Y^2 - (\mu_1 - \mu_2)^2.$$

Substituting

$$Y = \frac{1}{4} \frac{(\mu_1 - \mu_2)^{\frac{3}{2}} \eta}{(\xi - \mu_2)(\xi - \mu_1) \sqrt{\eta^2 (\mu_2 - \mu_1) - 8(\xi - \mu_1)(\xi - \mu_2)}}$$

into (4.30), We obtain the cubic in variables (ξ, η) with a full set of ten independent parameters. It easy to see that $\eta = a(x, y)\xi + b(x, y)$, where a, b - some functions.

Therefore in the cases of Class 2 and 3 the algebraic curve is non-hyperelliptic covering over the elliptic curve. The dynamics of the points $(\xi_1, Y_1), (\xi_2, Y_2), (\xi_3, Y_3)$ on this curve (see. theorem 2) defines the following condition: their projection on the elliptic base $(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)$ lies on the straight line.

Hypothesis 2. Any pair of the Hamiltonians (1.3)-(1.7) belongs to one of above three classes.

5 Appendix 1. Steklov top

We show that the case of Steklov top on $so(4)$ is a particular case of Class 1 after restriction on the symplectic leafs. Hamiltonian and the additional integral in this case have the form

$$H = (\vec{S}_1, A\vec{S}_1) + (\vec{S}_1, B\vec{S}_2), \quad K = (\vec{S}_1, \bar{A}\vec{S}_1) + (\vec{S}_1, \bar{B}\vec{S}_2),$$

where

$$A = -\alpha^2 \text{diag}\left(\frac{1}{\alpha_1^2}, \frac{1}{\alpha_2^2}, \frac{1}{\alpha_3^2}\right), \quad B = \alpha \text{diag}(\alpha_1, \alpha_2, \alpha_3),$$

$$\bar{A} = -\text{diag}(\alpha_1^2, \alpha_2^2, \alpha_3^2), \quad \bar{B} = \alpha \text{diag}\left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}\right),$$

$\alpha = \alpha_1 \alpha_2 \alpha_3$. Here \vec{S}_i - three-dimensional vectors with components S_i^α . It is easy to see that, H and K commute under a spin Poisson bracket

$$\{S_i^\alpha, S_j^\beta\} = \kappa \varepsilon_{\alpha\beta\gamma} \delta_{ij} S_i^\gamma.$$

It is convenient to chose $\kappa = -2i$.

We fix the Casimirs for spin bracket: $(\vec{S}_k, \vec{S}_k) = j_k^2$, $k = 1, 2$. Then the formulas

$$\vec{S}_k = \pi_k \vec{K}(Q_k) + \frac{j_k}{2} \vec{K}'(Q_k), \quad \text{где} \quad \vec{K}(Q) = ((Q^2 - 1), i(Q^2 + 1), 2Q),$$

define the Darboux coordinate π_1, π_2, Q_1, Q_2 for the symplectic leaf of Poisson manifold with coordinates \vec{S}_k , $k = 1, 2$. As this transformation is linear by momenta π_k , as a result, we obtain a pair of commuting Hamiltonians quadratic in momenta under the bracket $\{\pi_\alpha, Q_\beta\} = \delta_{\alpha\beta}$. Consider the canonical transformation that transforms their pair to the form (1.3)-(1.5), (4.22).

We apply the canonical transformation

$$P_1 = \pi_1 \sqrt{r(Q_1)}, \quad P_2 = \pi_2 \sqrt{R(Q_2)}, \quad dX = \frac{dQ_1}{\sqrt{r(Q_1)}}, \quad dY = \frac{dQ_2}{\sqrt{R(Q_2)}},$$

where

$$r(Q_1) = (\vec{K}(Q_1), A\vec{K}(Q_1)), \quad R(Q_2) = (\vec{K}(Q_2), \bar{A}\vec{K}(Q_2)),$$

to obtain

$$H = P_1^2 + 2P_1 P_2 V + j_2 P_1 V_Y + j_1 P_2 V_X + \frac{1}{2} j_1 j_2 V_{X,Y} + \frac{j_1^2}{6} \left(\frac{g_1''(X)}{g_1(X)} - \frac{3}{2} \left(\frac{g_1'(X)}{g_1(X)} \right)^2 \right),$$

$$K = P_2^2 + 2P_1 P_2 W + j_2 P_1 W_Y + j_1 P_2 W_X + \frac{1}{2} j_1 j_2 W_{X,Y} + \frac{j_2^2}{6} \left(\frac{g_2''(Y)}{g_2(Y)} - \frac{3}{2} \left(\frac{g_2'(Y)}{g_2(Y)} \right)^2 \right).$$

where

$$V(X, Y) = \frac{(\vec{K}(Q_1), B\vec{K}(Q_2))}{\sqrt{r(Q_1)} \sqrt{R(Q_2)}}, \quad W(X, Y) = \frac{(\vec{K}(Q_1), \bar{B}\vec{K}(Q_2))}{\sqrt{r(Q_1)} \sqrt{R(Q_2)}},$$

$$g_1(X) = \sqrt{r(Q_1)}, \quad g_2(Y) = \sqrt{R(Q_2)}.$$

We apply the canonical transformation $(P_1, P_2, X, Y) \rightarrow (p_1, p_2, x, y)$ of the form

$$dX = \frac{1}{2} \left(\frac{dx}{\sqrt{S(x)}} + \frac{dy}{\sqrt{yS(y)}} \right), \quad dY = -\frac{1}{2} \left(\frac{dx}{\sqrt{xS(x)}} + \frac{dy}{\sqrt{S(y)}} \right),$$

$$P_1 = \frac{2}{\sqrt{x} - \sqrt{y}} \left[\left(p_1 - \frac{j_1 + j_2}{4(x - y)} \sqrt{\frac{y}{x}} \right) \sqrt{S(x)} - \left(p_2 + \frac{j_1 + j_2}{4(x - y)} \sqrt{\frac{x}{y}} \right) \sqrt{S(y)} \right],$$

$$P_2 = \frac{2}{\sqrt{x} - \sqrt{y}} \left[\left(p_1 - \frac{j_1 + j_2}{4(x-y)} \sqrt{\frac{y}{x}} \right) \sqrt{y S(x)} - \left(p_2 + \frac{j_1 + j_2}{4(x-y)} \sqrt{\frac{x}{y}} \right) \sqrt{x S(y)} \right],$$

to obtain (1.3)-(1.5), (4.22), where

$$S(x) = -4x(1 + \alpha_1^2 x)(1 + \alpha_2^2 x)(1 + \alpha_3^2 x), \quad Z(x, y) = -\frac{1}{2}j_1(x + y) - j_2\sqrt{xy},$$

$$f(x) = \frac{1}{4} \left(j_1^2 \alpha^2 x^2 + \frac{j_2^2}{x} \right) - j_2^2 \frac{1}{4} \alpha^2 \left(\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} \right) x.$$

Acknowledgements. The author are grateful to E V Ferapontov for useful discussion. The research was partially supported by RFBR grant 05-01-00189 and NSh 6358.2006.2.

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